SOLVABILITY OF MULTIPOINT BVPs AT RESONANCE FOR VARIOUS KERNELS

Phan Dinh Phung

Ho Chi Minh City University of Food Industry

Email: *pdphungvn@gmail.com*

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ABSTRACT

In this paper, the Mawhin's continuation theorem in the theory of coincidence degree has been used to investigate the existence of solutions for a class of nonlinear second-order differential systems of equations in $\mathbb{R}^n$ associated with a multipoint boundary conditions at resonance. This is the first time that a resonant boundary condition of multipoint with large dimension of the kernel has been considered. An example has also been provided to illustrate the result.

Keywords: Coincidence degree, Fredholm operator of index zero, multipoint boundary value problem, resonance.

1. INTRODUCTION

In the theory of partial differential equations such as the method of separation of variables, we encounter differential equations for several parameters with some requirement of solutions which is called multi-point boundary condition. This then leads to an extensive development of spectral theory with multi-parameter [1]. Many multi-point boundary value problems (for short, BVPs) are established when looking for solutions to free-boundary problems [2]. Multi-point BVPs can also arise in other ways like physics and mechanics [3, 4]. In recent decades, the nonlinear multi-point BVPs especially at resonance have received much attention of many mathematicians, for instance, with the results of higher order BVPs [5, 6], the fractional order BVPs [7, 8], the positive solutions [9]. In particular, Phung P.D. *et al.* also had some contributions on this topic [10-12].

This note is to study the existence of solutions to the *m*-point BVPs in $\mathbb{R}^n$

$$\begin{cases} u''(t) = f(t, u(t), u'(t)), t \in (0,1), \\ u'(\theta) = 0, u(1) = \sum_{i=1}^{m-2} A_i u(\eta_i), \end{cases} \tag{1.1}$$

where θ is zero element in $\mathbb{R}^n$, $f: \mathbb{R}^{2n} \to \mathbb{R}^n$ satisfies the Carathéodory condition, that is,

(a) $f(\cdot, u, v)$ is Lebesgue measurable for every $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$,
(b) $f(t,\cdot,\cdot)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n$ for almost every $t \in [0, 1]$,
(c) for each compact set $K \subseteq \mathbb{R}^{2n}$, the function $h_K(t) = \sup\{|f(t, u, v)|: (u, v) \in K\}$

is Lebesgue integrable on $[0, 1]$, where $|\cdot|$ is the max-norm in $\mathbb{R}^n$, and $\eta_1, \eta_2, \dots, \eta_{m-2} \in (0,1), m \geq 3$, and $A_1, A_2, \dots, A_{m-2}$ are square matrices of order n satisfying

(G1) The matrix $I - \sum_{i=1}^{m-2} \eta_i A_i$ is invertible, (G2) $\left(\sum_{i=1}^{m-2}\eta^2{}_i A_i\right)\left(\sum_{i=1}^{m-2}A_i\right)=\left(\sum_{i=1}^{m-2}A_i\right)\left(\sum_{i=1}^{m-2}\eta^2{}_i A_i\right)$,

(G3) $\left(\sum_{i=1}^{m-2}A_i\right)^2=\sum_{i=1}^{m-2}A_i$ or $\left(\sum_{i=1}^{m-2}A_i\right)^2=I$, here I stands for the identity matrix of order n.

In most of boundary conditions, the operator $Lu = u''$ defined on some Banach spaces is invertible. Such a case is the so-called non-resonant; otherwise, the more complicated one called resonant.

In [13], Gupta first studied the existence of solutions for m-point BVP of the form

$$\begin{cases} u''(t) = f(t,u(t),u'(t)), t \in (0,1), \\ u'(\theta) = 0, u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \end{cases}$$

at resonance with the resonant condition $\sum_{i=1}^{m-2}\alpha_i = 1$. After that, Feng [14] and Ma [15] also achieved these similar results with some improvement on the nonlinear term. The main tool is Mawhin continuation theorem. This strongly depends on the dimension of ker L and most of results consider only the case that dim ker $L = 1$, because of which constructing projections P and Q (in Mawhin's method) is quite simple. Therefore, the larger dimension of ker L is, the more difficult the resonant problem is.

In this paper, we aim to generalize these works so that considering the multi-point BVPs (1.1) with the difficulty of resonance that $1 \le \dim\ker L \le n$ by using the Mawhin's continuation theorem. In addition, an example to illustrate the main result, especially the resonant conditions, was provided.

2. PRELIMINARIES

We begin this section by recalling some definitions and abstract results from the coincidence degree theory. For more details on the Mawhin's theory, we refer to [16, 17]. Let X and Z be two Banach spaces.

Definition 2.1. ([Ch. III-16, 17]) Let $L: domL \subset X \to Z$ be a linear operator. Then one says that L is a Fredholm operator provided that

(i) ker L is finite dimensional,

(ii) Im L is closed and has finite codimension.

Then the index of L is defined by

$$\operatorname{ind} L = \dim\ker L - \operatorname{codim}\operatorname{Im} L.$$

It follows from Definition 2.1 that if L is a Fredholm operator of index zero then there exist continuous projections $P: X \to X$ and $Q: Z \to Z$ such that

$$\operatorname{Im} P = \ker L,\ \ker Q = \operatorname{Im} L,\ X = \ker L \oplus \ker P,\ Z = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

Furthermore, the restriction of L on $domL \cap \ker P$, $L_P: domL \cap \ker P, \to \operatorname{Im} L,$ is invertible. We will denote its inverse by K_P. The generalized inverse of L denoted by $K_{P,Q} = K_P(I-Q)$. On the other hand, for every isomorphism $J: \operatorname{Im} Q \to \ker L,$ the mapping $JQ + K_{P,Q}: Z \to domL$ is an isomorphism, and

$$\left(JQ+K_{P,Q}\right)^{-1}u=\left(L+J^{-1}P\right)u,\ \forall u\in domL.$$

Now let Ω be an open bounded subset of X such that $domL\cap\Omega\neq\emptyset.$

Definition 2.2. ([Ch. III-16, 17]) Let L be a Fredholm operator of index zero. The operator $N:X\to Z$ is said to be L-compact in Ω if

- the map $QN:\overline{\Omega}\to Z$ is continuous and $QN\left(\overline{\Omega}\right)$ is bounded in Z,
- the map $K_{P,Q}N:\overline{\Omega}\to X$ is completely continuous.

Moreover, we say that *N* is *L*-completely continuous if it is *L*-compact on every bounded set in *X*.

Note that if *L* is a Fredholm operator of index zero and *N* is *L*-compact in $\overline{\Omega}$ then the existence of a solution to equation $Lu=Nu,\ u\in\overline{\Omega}$ is equivalent to the existence of a fixed point of Φ in $\overline{\Omega}$, where

$$\Phi=P+\left(JQ+K_{P,Q}\right)N.$$

This can be guaranteed by the following theorem due to Mawhin [16].

Theorem 2.1. *Let* $\Omega\in X$ *be open and bounded, L be a Fredholm mapping of index zero and N be L-compact on* $\overline{\Omega}$. *Assume that the following conditions are satisfied:*

i) $Lu\neq\lambda Nu$ *for every* $(u,\lambda)\in\left(\left(domL\setminus\ker L\right)\cap\partial\Omega\right)\times(0,1);$

ii) $QNu\neq 0$ *for every* $u\in\ker L\cap\partial\Omega;$

iii) for some isomorphism $J:\operatorname{Im}Q\to\ker L$ *we have*

$$\deg_B\left(JQN|_{\ker L};\Omega\cap\ker L,\theta\right)\neq 0,$$

where $Q:Z\to Z$ *is a projection given as above.*

Then the equation $Lu=Nu$ *has at least one solution in* $domL\cap\overline{\Omega}$.

Next, to achieve the existence of problem (1.1) by applying Theorem 2.1, we introduce the spaces $X=C^1\left([0,1];\mathrm{R}^n\right)$ endowed with the norm

$$\|u\|=\max\left\{\|u\|_\infty,\|u'\|_\infty\right\}$$

where $\|.\|_\infty$ stands for the sup-norm and $Z=L^1\left([0,1];\mathrm{R}^n\right)$ endowed with the Lebesgue norm denoted by $\|.\|_1$. Further, we use the Sobolev space defined by

$$X_0=\left\{u\in X:u''\in Z\right\}\subset X.$$

Then we define the operator $L:domL\subset X\to Z$ by $Lu=u''$, where

$$domL=\left\{u\in X_0:u'(0)=\theta,u(1)=\sum_{i=1}^{m-2}A_i\left(\eta_i\right)\right\}.$$

It is easy to see that $u\in X_0\Leftrightarrow u(t)=u(0)+u'(0)t+I_{0^+}^2Lu(t)$, where

$$I_{0^+}^k z(t)=\int_0^t(t-s)^{k-1}z(s)ds,\ \text{for } k\in\{1,2\}.$$

Thus, by substituting the boundary conditions, $\operatorname{dom}L$ is easily written by

$$domL = \left\{u \in X_0 : u(t) = u(0) + I_{0^+}^2 Lu(t) \text{ with } \mathrm{M}u(0) = \phi(Lu)\right\}, \tag{2.1}$$

where

- $\mathrm{M} = I - \sum_{i=1}^{m-2} A_i$
- $\phi : Z \to R^n$ is a continuous linear mapping defined by

$$\phi(z) = \sum_{i=1}^{m-2} A_i I_{o^+}^2 z(\eta_i) - I_{o^+}^2 z(1),\ z \in Z. \tag{2.2}$$

Hence, it is not difficult to show that

$$\ker L = \left\{u \in X : u(t) = c,\ t \in [0,1],\ c \in \ker M\right\} \cong \ker M.$$

Moreover we have

$$\operatorname{Im} L = \left\{z \in Z : \phi(z) \in \operatorname{Im} M\right\}.$$

Indeed, let $z \in \operatorname{Im} L$ so that $z = Lu$ for some $u \in domL.$ From (2.1) we have

$$Mu(0) = \phi(z)$$

which implies $\phi(z) \in \operatorname{Im} M$. Conversely, if $z \in Z$ such that $\phi(z) = M\xi \in \operatorname{Im} M$ then it is easy to see that $z = Lu$, where $u \in domL,$ defined by

$$u(t) = \xi + I_{o^+}^2 z(t).$$

This shows that $z \in \operatorname{Im} L.$

Now we prove some useful lemmas. The methods of the proofs are similar to some previous works [6, 10-12].

Lemma 2.1. *Assume that (G1)-(G3) hold. Then the operator L is a Fredholm operator of index zero.*

Proof. Since ϕ is continuous and $\operatorname{Im} M$ is closed in R^n it is clear that $\operatorname{Im} L$ is a closed subspace of Z. Further, we have $\dim\ker L = \dim\ker M \le n < \infty.$ It remains to show that $\operatorname{codim}\operatorname{Im} L = \dim\ker L.$ For this we consider the continuous linear mapping $Q : Z \to Z$ defined by, for $z \in Z,$

$$Qz(t) = (I - \kappa M) D\phi(z),\ t \in [0,1] \tag{2.3}$$

where

$$\kappa = \begin{cases} 1, \ \textit{if}\ (G3)_1 \ \textit{holds, that is,} \ \left(\sum_{i=1}^{m-2} A_i\right)^2 = \sum_{i=1}^{m-2} A_i, \\ \frac{1}{2}, \ \textit{if}\ (G3)_2 \ \textit{holds, that is,} \ \left(\sum_{i=1}^{m-2} A_i\right)^2 = I, \end{cases}$$

and

$$D = 2\left(\sum_{i=1}^{m-2} \eta_i^2 A_i - I\right)^{-1}.$$

Since (G1) holds, the matrix D exists. It's necessary to note that if $z(t) = h \in R^n,\ \ \forall t \in [0,1],$ then

$$\phi(z)=\sum_{i=1}^{m-2}A_i\int_0^{\eta_i}(\eta_i-s)hds-\int_0^1(1-s)hds=D^{-1}h. \tag{2.4}$$

From (G3), it's not difficult to show that κM and $I-\kappa M$ are two projections on R^n. Moreover, one can prove, for two cases, that

$$(I-\kappa M)D^{-1}=D^{-1}(I-\kappa M). \tag{2.5}$$

Indeed, if $(G3)_1$ holds then $I-\kappa M=\sum_{i=1}^{m-2}A_i$. By (G2) we get (2.5). Otherwise, we have $I-\kappa M=\frac{1}{2}\left(I+\sum_{i=1}^{m-2}A_i\right)$. Therefore

$$\left(I+\sum_{i=1}^{m-2}A_i\right)\left(\sum_{i=1}^{m-2}\eta_i^2A_i-I\right)=\sum_{i=1}^{m-2}\eta_i^2A_i\left(I+\sum_{i=1}^{m-2}A_i\right)-\left(I+\sum_{i=1}^{m-2}A_i\right)$$
$$=\left(\sum_{i=1}^{m-2}\eta_i^2A_i-I\right)\left(I+\sum_{i=1}^{m-2}A_i\right).$$

Hence (2.5) is proved. This follows from (2.4) – (2.5) that

$$Q^2z(t)=(I-\kappa M)D\phi(Qz)=(I-\kappa M)DD^{-1}Qz=(I-\kappa M)Qz$$
$$=(I-\kappa M)^2z(t)=(I-\kappa M)z(t)=Qz(t),\ t\in[0,1].$$

Thus, the map Q is idempotent and consequently Q is a continuous projection. Now we prove the following three assertions

i) $\ker Q=\operatorname{Im}L,$

ii) $Z=\operatorname{Im}L\oplus\operatorname{Im}Q,$

iii) $\operatorname{Im}Q=\ker L,$

which allow us to complete the proof of the lemma. In order to get i) we note that $D^{-1}M=MD^{-1}$, due to (2.5), which implies $DM=MD.$ So $\alpha\in\operatorname{Im}M$ if and only if $D\alpha\in\operatorname{Im}M$. Hence, for $z\in Z,$

$$z\in\ker Q\Leftrightarrow D\phi(z)\in\ker(I-\kappa M)\Leftrightarrow D\phi(z)\in\operatorname{Im}(\kappa M)\Leftrightarrow\phi(z)\in\operatorname{Im}(M)\Leftrightarrow z\in\operatorname{Im}L,$$

which shows that $\ker Q=\operatorname{Im}L.$ Hence, we also obtain ii), that is, $Z=\ker Q\oplus\operatorname{Im}Q=\operatorname{Im}L\oplus\operatorname{Im}Q.$ Now, let $z\in\operatorname{Im}Q.$ Assume that $z=Qz_1$, for $z_1\in Z.$ Then we have

$$\kappa Mz(t)=\kappa M(I-\kappa M)D\phi(z_1)=\theta,\ \ t\in[0,1],$$

due to κM is a projection. This implies $Qz_1\in\ker(\kappa M)\equiv\ker M$. Therefore $z\in\ker L.$ Conversely, for each $z\in\ker L,$ there exists $\beta\in\ker M$ such that $z(t)=\beta$ for all $t\in[0,1].$ Then we have

$$Qz(t)=(I-\kappa M)D\phi(z)=(I-\kappa M)D\left(D^{-1}\beta\right)=(I-\kappa M)\beta=\beta=z(t),\ \ t\in[0,1],$$

Hence $z\in\operatorname{Im}Q$ and so we get $\operatorname{Im}Q=\ker L.$ Then Lemma 2.4 has proved.

Now we define an operator $P: X \to X$ by setting

$$Pu(t) = (I - \kappa M)u(0), \ \forall t \in [0,1]. \tag{2.6}$$

Lemma 2.2. *We have*

i) The mapping P defined by (2.6) is a continuous projection satisfying the identities

$$\operatorname{Im} P = \ker L, \ X = \ker L \oplus \ker P.$$

ii) The linear operator $K_P : \operatorname{Im} L \to domL \cap \ker P$ *can be defined by*

$$K_P z(t) = \kappa^2 M\phi(z) + I_{0^+}^2 z(t), \ t \in [0,1], \tag{2.7}$$

Moreover K_P *satisfies*

$$K_P = \left(L|_{domL \cap \ker P}\right)^{-1} \text{ and } \|K_P z\| \le C\|z\|_1,$$

where $C = 1 + \kappa^2 \|M\|_* \left(1 + \sum_{1=1}^{m-2} \eta_i \|A_i\|_*\right)$ ($\|\cdot\|_*$ *is the maximum absolute column sum norm of matrices).*

Proof. i) It is clear that P is a continuous projection. Further we have $\operatorname{Im} P = \ker L.$ Indeed, if $v \in \operatorname{Im} P$ then there exists $u \in X$ such that

$$v(t) = Pu(t) = (I - \kappa M)u(0), \ \forall t \in [0,1]. \tag{2.8}$$

Thus

$$\kappa M v(t) = \kappa M (I - \kappa M)u(0) = \theta$$

which implies that $v \in \ker L,$ by the definition of $\ker L.$ Conversely if $v \in \ker L$ then

$$v(t) = \xi \in \ker M, \ \forall t \in [0,1].$$

Then we deduce that

$$Pv(t) = (I - \kappa M)v(0) = (I - \kappa M)\xi = \xi = v(t), \ \forall t \in [0,1].$$

This shows that $v \in \operatorname{Im} P.$ Therefore we can conclude that $\operatorname{Im} P = \ker L$ and consequently $X = \ker L \oplus \ker P.$

ii) Let $z \in \operatorname{Im} L.$ Then we have $\phi(z) \in \operatorname{Im} M$ which implies that $\phi(z) = M\beta,$ where $\beta \in R^n.$ It follows from (2.6) and (2.7) that

$$PK_P z(t) = (I - \kappa M)K_P z(0) = \kappa^2 (I - \kappa M)M\phi(z) = \theta, \ \forall t \in [0,1].$$

Thus $K_P z \in \ker P.$ In addition, clearly $\phi(z) \in \operatorname{Im} M$ and κM is the projection, implying

$$M\kappa\phi(z) = \kappa M\phi(z) = \phi(z).$$

Then, it is easy to show that $K_P z \in domL.$ So K_P is well defined. On the other hand, if $u \in domL \cap \ker P$ then $u(t) = u(0) + I_{0^+}^2 Lu(t)$, with

$$\begin{cases} Mu(0) = \phi(Lu), \\ u(0) \in \operatorname{Im}(\kappa M). \end{cases}$$

Thus

$$K_P Lu(t) = \kappa^2 M\phi(Lu) + I_{0^+}^2 Lu(t) = (\kappa M)^2 u(0) + I_{0^+}^2 Lu(t) = u(0) + I_{0^+}^2 Lu(t) = u(t)$$

by κM is a projection. This deduces that $K_P = \left(L|_{domL\cap\ker P}\right)^{-1}$ by $LK_P z(t) = z(t),\ t \in [0,1]$, for all $z \in \operatorname{Im} L$. Finally, from the definition of K_P we have

$$(K_P z)'(t) = I_{0^+}^1 z(t),\ t \in [0,1]. \tag{2.9}$$

Combining (2.2), (2.7) and (2.9) we have

- $\|K_P z\|_\infty \le \kappa^2 \|M\|_* |\phi(z)| + \|z\|_1$,
- $|\phi(z)| \le \left(1 + \sum_{i=1}^{m-2} \eta_i \|A_i\|_*\right)\|z\|_1$,
- $\left\|(K_P z)'\right\|_\infty \le \|z\|_1$.

These show that $\|K_P z\| \le C\|z\|_1$. The lemma is proved.

Lemma 2.3. *The operator* $N : X \to Z$ *defined by*

$$Nu(t) = f(t, u(t), u'(t)),\ a.e.,\ t \in [0,1]$$

is L-completely continuous.

Proof. Let Ω be a bounded set in *X*. Put $R = \sup\{\|u\| : u \in \Omega\}$. From the assumptions of the function *f* there exists a function $m_R \in Z$ such that, for all $u \in \Omega$ we have

$$|Nu(t)| = |f(t, u(t), u'(t))| \le m_R(t),\ a.e.,\ t \in [0,1] \tag{2.10}$$

It follows from (2.2), (2.13) and the identity

$$QNu(t) = (I - \kappa M) D\phi(Nu) \tag{2.11}$$

that $QN(\overline{\Omega})$ is bounded and QN is continuous by using the Lebesgue's dominated convergence theorem. We now prove that $K_{P,Q}N$ is completely continuous. Note that, for every $u \in \Omega$, we have

$$\begin{aligned} K_{P,Q}Nu(t) &= K_P(I - Q)Nu(t) \\ &= K_P(Nu - QNu)(t) \\ &= K_P\left[Nu - (I - \kappa M)D\phi(Nu)\right](t) \\ &= I_{0^+}^2 Nu(t) - \frac{t^2}{2}(I - \kappa M)D\phi(Nu) + \kappa^2 M\phi(Nu), \end{aligned} \tag{2.12}$$

and

$$(K_{P,Q}Nu)'(t) = I_{0^+}^1 Nu(s)ds - t(I - \kappa M)D\phi(Nu). \tag{2.13}$$

Further, it follows from (2.13) and the definition of ϕ that

$$|\phi(z)| \le \left(1 + \sum_{i=1}^{m-2} \eta_i \|A_i\|_*\right)\|z\|_1 \le \left(1 + \sum_{i=1}^{m-2} \eta_i \|A_i\|_*\right)\|m_R\|_1, \tag{2.14}$$

Combining (2.10) and (2.12) – (2.14) we can find two positive constants $C_1,\ C_2$ such that

$$\left|K_{P,Q}Nu(t)\right| \le C_1 \left\|m_R\right\|_1,\ \left|\left(K_{P,Q}Nu\right)'(t)\right| \le C_2 \left\|m_R\right\|_1, \tag{2.15}$$

for all $t\in[0,1]$ and for all $u\in\Omega$. This shows that

$$\left\|K_{P,Q}Nu\right\| \le \max\{C_1, C_2\}\left\|m_R\right\|_1,$$

that is, $K_{P,Q}N(\Omega)$ is uniformly bounded in *X*. On the other hand, for $t_1,\ t_2\in[0,1]$ with $t_1<t_2,$ we have

$$\left|K_{P,Q}Nu(t_2)-K_{P,Q}Nu(t_1)\right| \le \int_{t_1}^{t_2} ds\int_0^a \left|Nu(\tau)\right|d\tau + \left|(t_2-t_1)(I-\kappa M)D\phi(Nu)\right|$$
$$\le C_3\left\|m_R\right\|_1 \left|t_2-t_1\right|,$$

and

$$\left|\left(K_{P,Q}Nu\right)'(t_2)-\left(K_{P,Q}Nu\right)'(t_1)\right| \le \int_{t_1}^{t_2} m_R(s)\,ds + C_4\left\|m_R\right\|_1\left|t_2-t_1\right|,$$

which prove that the family $K_{P,Q}N(\Omega)$ is equi-continuous in *X*. Thanks to Arzelà-Ascoli theorem, $K_{P,Q}N(\Omega)$ is a relatively compact subset in *X*. Lastly, it is obvious that $K_{P,Q}N$ is continuous. Therefore, the operator *N* is *L*-completely continuous. The proof of the theorem is completed.

3. MAIN RESULTS

In this section we employ Theorem 2.1 to prove the existence of the solutions of problem (1.1). For this aim we assume that the following conditions hold:

(B1) there exist the positive functions $a,\ b,\ c\in Z$ with $\left(\left\|I-\kappa M\right\|_* + C\right)\left(\left\|a\right\|_1+\left\|b\right\|_1\right)<1$ such that

$$\left|f(t,u,v)\right| \le a(t)|u| + b(t)|v| + c(t), \tag{3.1}$$

for all $t\in[0,1]$ and $u,\ v\in R^n$ where C is the constant given in Lemma 2.2;

(B2) there exists a positive constant Λ_1 such that for each $u\in domL,$ if $\left|u(t)\right|>\Lambda_1,\ \forall t\in[0,1],$ then

$$\sum_{i=1}^{m-2} A_i\int_{\eta_i}^{1} ds\int_0^s f\left(\tau,u(\tau),u'(\tau)\right)d\tau \notin \operatorname{Im} M; \tag{3.2}$$

(B3) there exists a positive constant Λ_2 such that for any $\alpha\in R^n$ with $\alpha\in\ker M$ and $|\alpha|>\Lambda_2$, either

$$\left\langle \alpha, QN(\alpha)\right\rangle \le 0 \text{ or } \left\langle \alpha, QN(\alpha)\right\rangle \ge 0, \tag{3.3}$$

where $\langle .,.\rangle$ stand for the scalar product in R^n.

Lemma 3.1. *Let* $\Omega_1 = \{u \in domL \setminus \ker L: \ Lu = \lambda Nu, \ \lambda \in [0,1]\}$. *Then* Ω_1 *is bounded in X.*

Proof. Let $u \in \Omega_1$. Assume that $Lu = \lambda Nu$ for $0 < \lambda \le 1$. Then it is clear that $Nu \in \operatorname{Im} L = \ker Q$, which implies $\phi(Nu) \in \operatorname{Im} M$ by the definition of $\operatorname{Im} L$. On the other hand, we have

$$\sum_{i=1}^{m-2} A_i \int_{\eta_i}^{1} ds \int_0^s f\left(t, u(\tau), u'(\tau)\right) d\tau = -\phi(Nu) - M \int_0^1 ds \int_0^s f\left(t, u(\tau), u'(\tau)\right) d\tau.$$

Hence we deduce that

$$\sum_{i=1}^{m-2} A_i \int_{\eta_i}^{1} ds \int_0^s f\left(\tau, u(\tau), u'(\tau)\right) d\tau \in \operatorname{Im} M.$$

By assumption (B2), there exists $t_0 \in [0,1]$ such that $|u(t_0)| \le \Lambda_1$. Then we get

$$|u(t)| = \left| u(t_0) + \int_{t_0}^{t} u'(s)\, ds \right| \le \Lambda_1 + \|u'\|_\infty \text{ and } |u'(t)| \le \int_0^t |u''(s)|\, ds \le \|u''\|_1 \le \|Nu\|_1, \tag{3.4}$$

for all $t \in [0,1]$. These give us

$$\|Pu\| = \left|(I - \kappa M) u(0)\right| \le \|I - \kappa M\|_* \left(\Lambda_1 + \|Nu\|_1\right). \tag{3.5}$$

On the other hand, it is noted that $(Id_X - P)u \in domL \cap \ker P$ since P is a projection on X. Then

$$\left\|(Id_X - P)u\right\| = \left\|K_P L (Id_X - P) u\right\| \le \|K_P L u\| \le C\|Nu\|_1, \tag{3.6}$$

where the constant C is defined as in Lemma 2.5 and Id_X is the identity operator on X. Combining (3.5) and (3.6) obtains

$$\|u\| = \left\|Pu + (Id_X - P)u\right\| \le \|Pu\| + \left\|(Id_X - P)u\right\| \le \Lambda_1 \|I - \kappa M\|_* + \left(\|I - \kappa M\|_* + C\right)\|Nu\|_1. \tag{3.7}$$

By (B1) and the definition of N we have

$$\|Nu\|_1 \le \int_0^1 \left| f\left(s, u(s), u'(s)\right)\right| ds \le \|a\|_1 \|u\|_\infty + \|b\|_1 \|u'\|_\infty + \|c\|_1 \le \left(\|a\|_1 + \|b\|\right)_1 \|u\| + \|c\|_1. \tag{3.8}$$

Combining (3.7) and (3.8) gives us

$$\|Nu\|_1 \le \frac{\Lambda_1 \|I - \kappa M\|_* \left(\|a\|_1 + \|b\|_1\right) + \|c\|_1}{1 - \left(\|I - \kappa M\|_* + C\right)\left(\|a\|_1 + \|b\|_1\right)}.$$

The last inequality and (3.4) deduce that

$$\sup_{u \in \Omega_1} \|u\| = \sup_{u \in \Omega_1} \max\left\{\|u\|_\infty, \|u'\|_\infty\right\} < +\infty.$$

Therefore Ω_1 is bounded in X. The lemma is proved.

Lemma 3.2. *The set* $\Omega_2 = \{u \in \ker L : Nu \in \operatorname{Im} L\}$ *is a bounded subset in X.*

Proof. Let $u \in \Omega_2$. Assume that $u(t) = c,\ \forall t \in [0,1]$, where $c \in \ker M$. Since $Nu \in \operatorname{Im} L$ we have $\phi(Nu) \in \operatorname{Im} M$. By the same arguments as in the proof of Lemma 3.1 we can point out that there exists $t_0 \in [0,1]$ such that $|u(t_0)| \le \Lambda_1$. Therefore

$$\|u\| = \|u\|_\infty = |u(t_0)| = |c| \le \Lambda_1.$$

So Ω_2 is bounded in *X*. The lemma is proved.

Lemma 3.3. *The sets* $\Omega_3^- = \{u \in \ker L : -\lambda u + (1-\lambda) QNu = \theta, \lambda \in [0,1]\}$ *and*

$$\Omega_3^+ = \{u \in \ker L : \lambda u + (1-\lambda) QNu = \theta, \lambda \in [0,1]\}$$

are bounded in X provided that the first and the second part of (3.3) is satisfied, respectively.

Proof. Case 1: $\langle \alpha, QN\alpha \rangle \le 0$. Let $u \in \Omega_3^-$. Then there exists $\alpha \in \ker M$ such that $u(t) = \alpha, \forall t \in [0,1]$, and

$$(1-\lambda) QN\alpha = \lambda\alpha. \tag{3.9}$$

If $\lambda = 0$ then it follows from (3.9) that $N\alpha \in \ker Q = \operatorname{Im} L$, which means $u \in \Omega_2$. Using Lemma 3.2 we deduce that $\|u\| \le \Lambda_1$. On the other hand, if $\lambda \in [0,1]$ and $|\alpha| > \Lambda_2$ then, by assumption (B3), we get a contradiction

$$0 < \lambda |\alpha|^2 = (1-\lambda) \langle \alpha, QN\alpha \rangle \le 0.$$

Thus $\|u\| = |\alpha| \le \Lambda_2$ or Ω_3^- is bounded in X .

Case 2: $\langle \alpha, QN\alpha \rangle \ge 0$. In this case, using the similar arguments as in **Case 1** we show that Ω_3^+ is also bounded in *X*.

Theorem 3.1. *Let the assumptions (B1)-(B3) hold. Then the problem (1.1) has at least one solution in X.*

Proof. We prove that all of the conditions of Theorem 2.1 are satisfied, where Ω be open and bounded such that $\bigcup_{i=1}^3 \overline{\Omega}_i \subset \Omega.$. It is clear that the conditions (1) and (2) of Theorem 2.1 are fulfilled by using Lemma 3.1 and Lemma 3.2. So, it remains to verify that the third condition holds. For this, we apply the degree property of invariance under a homotopy. Let us define $H(u,\lambda) = \pm\lambda u + (1-\lambda) QNu$, where we choose the isomorphism $J : \operatorname{Im} Q \to \ker L$ is the identity operator. By Lemma 3.3, we have $H(u,\lambda) \ne \theta$ for all $(u,\lambda) \in (\ker L \cap \partial\Omega) \times [0,1]$, so that

$$\begin{aligned}\deg\left(QN|_{\ker L}; \Omega \cap \ker L, \theta\right) &= \deg\left(H(.,0), \Omega \cap \ker L, \theta\right) = \deg\left(H(.,1), \Omega \cap \ker L, \theta\right) \\ &= \deg\left(\pm Id, \Omega \cap \ker L, \theta\right) = \pm 1 \ne 0.\end{aligned}$$

Hence, Theorem 3.1 is proved.

In order to end this paper, we provide an example dealing with the solvability of a second order system of differential equations associated with four-point boundary conditions by applying the above results.

Example 3.1. Consider the following boundary value problem

$$\begin{cases} x''(t) = f_1(t, x(t), y(t), x'(t), y'(t)), t \in (0,1), \\ y''(t) = f_2(t, x(t), y(t), x'(t), y'(t)), t \in (0,1), \\ x'(0) = y'(0) = 0, \\ x(1) = -4x(1/3) + 2y(1/3) + 6x(1/2) - 4y(1/2), \\ y(1) = -x(1/3) - y(1/3) + 2x(1/2), \end{cases} \tag{3.10}$$

where the functions $f_i : [0,1] \times R^4 \to R \ (i = 1,\ 2)$ are given by

$$f_1(t, x_1, y_1, x_2, y_2) = \frac{t+2}{180\sqrt{2}}(x_1 + y_1) + \frac{t^5}{60}\ln\left(1 + \sqrt{x_2^2 + y_2^2}\right), \tag{3.11}$$

$$f_2(t, x_1, y_1, x_2, y_2) = \frac{t+2}{180\sqrt{2}}(|x_1| + |y_1|) + \frac{t^5}{60}\sqrt{x_2^2 + y_2^2}, \tag{3.12}$$

for all $t \in [0,1]$ and $(x_1, y_1),\ (x_2, y_2) \in R^2$.

In what follows we prove that problem (3.10) has at least one solution by using Theorem 3.1. First we put

$$\eta_1 = 1/3,\ \eta_2 = 1/2,\ A_1 = \begin{bmatrix} -4 & 2 \\ -1 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} 6 & -4 \\ 2 & 0 \end{bmatrix}$$

and define the function $f : [0,1] \times R^2 \times R^2 \to R^2$ by

$$f(t, u_1, u_2) = (f_1(t, u_1, u_2),\ f_2(t, u_1, u_2)), \tag{3.13}$$

for all $t \in [0,1]$ and $u_1 = (x_1, y_1),\ u_2 = (x_2, y_2) \in R^2$. Then problem (3.10) has one solution if and only if problem (1.1) (with $m = 4,\ \eta_1,\ \eta_2,\ A_1,\ A_2$ and f defined as above) has one solution. So we need only show that the conditions of Theorem 3.1 hold.

Indeed, it is clear that $I - (\eta_1^2 A_1 + \eta_2^2 A_2)$ is invertible,

$$(\eta_1^2 A_1 + \eta_2^2 A_2)(A_1 + A_2) = (A_1 + A_2)(\eta_1^2 A_1 + \eta_2^2 A_2) = \begin{bmatrix} 4/3 & -4/3 \\ 2/3 & -2/3 \end{bmatrix},$$

and

$$(A_1 + A_2)^2 = A_1 + A_2 = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$

On the other hand, from (3.11) - (3.13), the function f satisfies Carathéodory condition. Next we verify conditions (B1) - (B3). It follows from (3.11), (3.12) and (3.13) that

$$|f(t, u, v)| \le a(t)|u| + b(t)|v|,$$

for all $t \in [0,1]$ and $u,\ v \in R^2$, where

$$a(t) = \frac{t+2}{90}, b(t) = \frac{t^5}{30}.$$

Since $a,\ b \in L^1([0,1]; \mathrm{R}^+)$ and $(\|I - \kappa M\|_* + C)(\|a\|_1 + \|b\|_1) = 41/90 < 1$, condition (B1) is satisfied. In order to check (B2) we note that

$$f_1(t,u(t),u'(t)) < f_2(t,u(t),u'(t)),\ t\in[0,1],$$

for all $u=(x,y)\in dom(L)$. This implies that

$$\sum_{i=1}^{m-2} A_i \int_{\eta_i}^{1} ds \int_0^s f_1(t,u(t),u'(t))\,dt \neq \sum_{i=1}^{m-2} A_i \int_{\eta_i}^{1} ds \int_0^s f_2(t,u(t),u'(t))\,dt.$$

Therefore

$$\sum_{i=1}^{m-2} A_i \int_{\eta_i}^{1} ds \int_0^s f(t,u(t),u'(t))\,dt \notin \operatorname{Im}(M)$$

due to $\operatorname{Im}(M)=\{(q,q): q\in R\}$. It means that (B2) holds. Finally, we note that

$$D=D_G=\begin{bmatrix} -120/13 & 84/13 \\ -42/13 & 6/13 \end{bmatrix}.$$

Then

$$Qz(t)=(I-\kappa M)D\phi(z)=\begin{bmatrix} -12 & 12 \\ -6 & 6 \end{bmatrix}\phi(z),$$

for all $z\in L^1([0,1];\mathrm{R}^2)$, where

$$\phi(z)=A_1\int_0^{1/3} ds\int_0^s z(\tau)\,d\tau + A_2\int_0^{1/2} ds\int_0^s z(\tau)\,d\tau - \int_0^1 ds\int_0^s z(\tau)\,d\tau.$$

Let $\alpha=(2a,a)\in\ker(M)$, we have

$$N\alpha=(f_1(t,a,0),\, f_2(t,a,0))=\frac{t+2}{60\sqrt{2}}(a,|a|),$$

and

$$\phi(N\alpha)=\left(\frac{-7\sqrt{2}}{77760}a-\frac{55\sqrt{2}}{7776}|a|,\frac{55\sqrt{2}}{15552}a-\frac{13\sqrt{2}}{1215}|a|\right).$$

So we obtain

$$Q(N\alpha)=\frac{47\sqrt{2}}{2160}\left[2(a-|a|),a-|a|\right].$$

Thus,

$$\langle \alpha, QN\alpha\rangle=\frac{47}{432}\sqrt{2}\left(a^2-a|a|\right).$$

This shows $\langle \alpha, QN\alpha\rangle \geq 0$ for all $a\in R$, that means (B3) is satisfied. Thanks to Theorem 3.4, the problem (3.10) has at least one solution.

4. CONCLUSION

This note is dedicated to deal with the existence of a multi-point BVP for second-order differential systems at resonance in the case of various kernel spaces. This provides a technique to construct the two projections in the method of Mawhin's coincidence degree when dimension of the kernel is large as well as to prove an operator with complicated boundary condition is Fredholm of index zero.

In a forthcoming research, it is possible to extend to a wider class of resonant conditions on the matrices A_i and with a more general boundary condition.

REFERENCES

1. Gregus M., Neuman F., Arscott F. M. - Three-point boundary value problems in differential equations, Journal of the London Mathematical Society **s2-3** (3) (1971) 429-436.
2. Berger M. S., Fraenkel L. E. - Nonlinear desingularization in certain free-boundary problems, Communications in Mathematical Physics **77** (2) (1980) 149-172.
3. Moshiinsky M. - Sobre los problemas de condiciones a la frontiera en una dimension de caracteristicas discontinuas, Boletin De La Sociedad Matematica Mexicana **7** (1950) 1-25.
4. Timoshenko S. - Theory of Elastic Stability, McGraw-Hill, New York (1961).
5. Chang S. K., Pei M. - Solvability for some higher order multi-point boundary value problems at resonance, Results in Mathematics **63** (3) (2013) 763-777.
6. Lu S., Ge W. - On the existence of *m*-point boundary value problem at resonance for higher order differential equation, Journal of Mathematical Analysis and Applications **287** (2003) 522-539.
7. Jiang W. - The existence of solutions for boundary value problems of fractional differential equations at resonance, Nonlinear Analysis **74** (2011) 1987-1994.
8. Zhou H. C., Ge F. D., Kou C. H. - Existence of solutions to fractional differential equations with multi-point boundary conditions at resonance in Hilbert spaces, Electronic Journal of Differential Equations **61** (2016) 1-16.
9. Han X. - Positive solutions for a three-point boundary-value problem at resonance, J. Math. Anal. Appl. **336** (2007) 556-568.
10. Phung P. D., Truong L. X. - On the existence of a three point boundary value problem at resonance in R^n, Journal of Mathematical Analysis and Applications **416** (2014) 522-533.
11. Phung P. D., Truong L. X. - Existence of solutions to three-point boundary-value problems at resonance, Electronic Journal of Differential Equations **115** (2016) 1-13.
12. Phung P. D., Minh H. B. - Existence of solutions to fractional boundary value problems at resonance in Hilbert spaces, Boundary Value Problems (2017) 2017:105.
13. Gupta C. P. - Existence theorems for a second order *m*-point boundary value problem at resonance, International Journal of Mathematics and Mathematical Sciences **18** (4) (1995) 705-710.
14. Feng W. - On an *m*-point boundary value problems, Nonlinear Analysis **30** (1997) 5369-5374.
15. Ma R. - Existence results of a *m*-point boundary value problem at resonance, Journal of Mathematical Analysis and Applications **294** (2004) 147-157.
16. Gaines R. E., Mawhin J. - Coincidence Degree and Nonlinear Differential Equations, Lecture Notes in Math., vol. 568, Springer-Verlag, Berlin, 1977.

17. Mawhin J. - Theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces, Journal of Differential Equations **12** (1972) 610-636.

TÓM TẮT

TÍNH GIẢI ĐƯỢC CỦA BÀI TOÁN BIÊN ĐA ĐIỂM CỘNG HƯỞNG
VỚI NHÂN THAY ĐỔI

Phan Đình Phùng
Trường Đại học Công nghiệp Thực phẩm TP.HCM
Email: *pdphungvn@gmail.com*

Trong bài báo này tác giả sử dụng định lý liên tục Mawhin trong lý thuyết bậc coincidence để nghiên cứu sự tồn tại nghiệm cho một lớp hệ phương trình vi phân cấp hai phi tuyến trong $\mathbb{R}^n$ kết hợp với biên đa điểm trong điều kiện cộng hưởng. Đây là kết quả đầu tiên nghiên cứu điều kiện biên cộng hưởng loại đa điểm với số chiều của hạt nhân lớn. Tác giả cũng xây dựng một ví dụ để minh họa kết quả.

Từ khóa: Bậc coincidence, toán tử Fredholm chỉ số 0, bài toán biên đa điểm, cộng hưởng.